A Brief Introduction to
Resonance in the Damped, Driven Simple Harmonic Oscillator

just a small part of what can be done with TeachSpin's Torsional Oscillator

The simple harmonic oscillator is one of the models that are very generally applicable in all branches of physics. Students encounter it first in the case of one-dimensional motion under the action of a Hooke's-Law spring:

$$m\,a = F_{net} = -k\,x \qquad \text{gives} \qquad m\,d^2x/dt^2 + k\,x(t) = 0\ .$$

This is already enough to describe simple harmonic motion, and to predict sinusoids of period $T = 2\pi\sqrt{(m/k)}$. But of course there's lots more physics to be displayed: the net force could also include, in addition to the Hooke's-Law restoring force, a damping force and/or an external driving force.

TeachSpin has built a system in which every feature of damped, driven, simple harmonic motion can be understood qualitatively and measured quantitatively. The 'Torsional Oscillator' that has resulted uses one-dimensional rotational motion, of a rigid body or 'rotor' about a vertical axis, described by single angular coordinate $\theta(t)$. The equivalent of Newton's Second Law for this system is

$$I\,d^2\theta/dt^2 = \tau_{net}\ ,$$

where I is the rotor's rotational inertia, and τ_{net} is the net torque applied to the system.

In the apparatus, there are three main sources of torque:

- The system has a linear torsional restoring torque, provided by the twisting or torsion of a steel piano-wire fiber, so one term in τ_{net} is $\tau = -\kappa\,\theta$, where κ is the torsion constant of the fiber.
- The system also provides a means to give a damping torque (via eddy-current damping). This torque, very accurately proportional to, and opposite to, the angular velocity, is described by $\tau = -b\,d\theta/dt$.
- And the system also allows an external electric current $i(t)$ to create a driving torque, via a magnetic interaction. Current i produces magnetic field $B = k\,i$ via a set of coils, and that field B acts on a magnetic-dipole, μ, mounted on the rotor, to give a driving torque $\tau(t) = \mu\,B(t) = \mu\,k\,i(t)$.

So the result is a system described by

$$I\,d^2\theta/dt^2 = \tau_{net} = -\kappa\,\theta - b\,d\theta/dt + \mu\,k\,i(t)\ .$$

This can be re-arranged to give a standard form of differential equation,

$$I\,d^2\theta/dt^2 + b\,d\theta/dt + \kappa\,\theta(t) = \mu\,k\,i(t)\ .$$

This is the textbook equation for damped, driven, simple harmonic motion. Every constant in the equation stands for a parameter of the experimental system which can be measured by (one or more) direct experiments.

One limiting case of this general equation is that of freely decaying oscillations, created by an initial displacement with no subsequent external driving torque. The solution is predicted to be an exponentially-decaying sinusoid, with

$$\theta(t) \propto \theta_0 \exp(-\gamma\, \omega_0\, t) \cos[\omega_0\, t\sqrt{(1-\gamma^2)} - \varphi] \;,$$

where the 'natural frequency' ω_0 and the dimensionless damping parameter γ can both be related to the constants I, b, and κ :

$$\omega_0 = \sqrt{(\kappa/I)} \quad \text{and} \quad \gamma = b / [2 \sqrt{(\kappa I)}\,] \;.$$

Experimental data for $\theta(t)$ in such a 'hold and release' experiment is easily acquired. The oscillator is equipped with a sensitive angular-position transducer -- it produces a real-time voltage V(t) accurately proportional to $\theta(t)$.

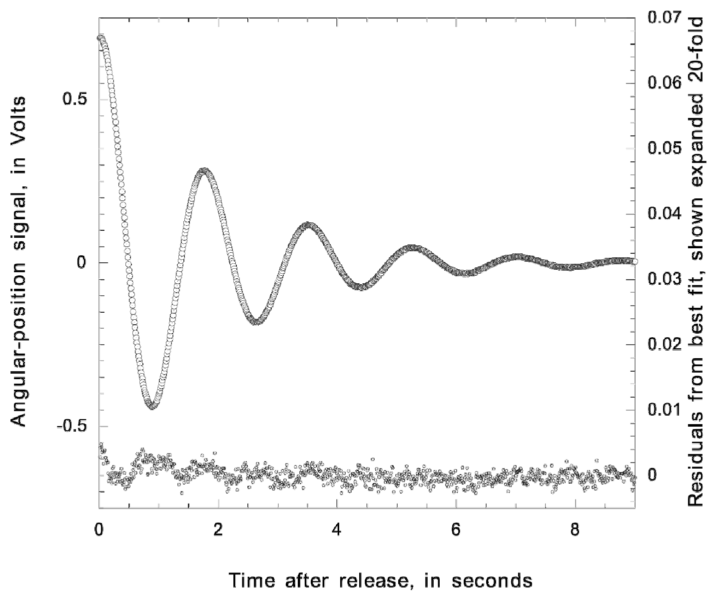

Fig. 1: The angular-position signal as a function of time, following release from a non-equilibrium position. At the bottom, on an expanded scale, the residuals remaining after fitting a decaying sinusoid to the data.

With such lovely data in hand, it's possible to best-fit the data to an equation of the form

$$\theta(t) = \theta_0 \exp(-\lambda\, t) \cos(\omega_d\, t - \text{const})$$

where λ is a damping rate, and where ω_d gives the frequency of the (damped) oscillations. For example, the data above give damping $\lambda = (0.5074 \pm 0.0002)\ s^{-1}$, and damped frequency $\omega_d = (3.5901 \pm 0.0002)\ s^{-1}$.

But it's easy to change the amount of damping and repeat this hold-and-release experiment, to get another pair of (λ, ω_d) values. One motivation is to test the prediction of the theory above that the damped frequency is reduced below the natural frequency ω_0. Sure enough, it's feasible to show experimentally that the frequency of oscillations, in the presence of damping, really is reduced by the damping, and the effect can be followed all the way to critical damping (where $\gamma = 1$).

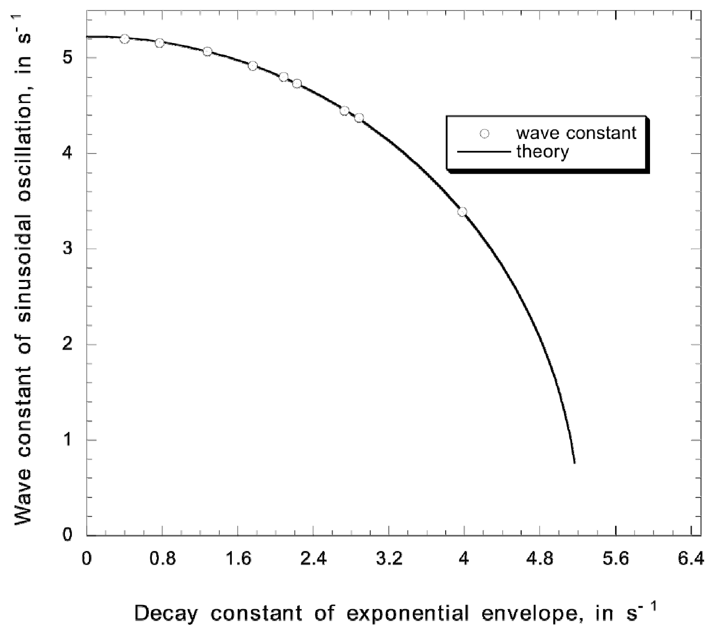

Fig. 2: Results derived from fits to transient-oscillation data, plotted as (x, y) pairs of (decay constant λ, wave constant ω_d) for various settings of the damping in the Torsional Oscillator. The solid line is the circular locus relating these numbers, as predicted by theory.

So a torsional oscillator can be fully characterized by various calibrations, and by this sort of hold-and-release experiment. But now it's time to change from calibrations (using static torques), or hold-and-release experiments (involving, for $t>0$, *zero* external torque), to the use of some arbitrary, continuing, time-dependent external torque. The most famous qualitative prediction about the expected result ins captured the intuition you gained on a playground swing: to 'pump up' an oscillator, you want to supply a drive that's periodic in time, with a period *matching* the oscillator's 'natural period'.

The simplest experiments demonstrating this quantitatively are those involving an ongoing sinusoidal drive current,

$$i(t) = I_0 \cos(\omega t) \ ,$$

with an amplitude I_0 and a frequency ω that can be freely chosen by settings on some external waveform generator. The response to this drive is predicted to settle (after some transients -- themselves worthwhile objects of study) into a steady state, also of sinusoidal oscillations:

$$\theta(t) = A \cos(\omega t - \phi) \ .$$

Here A is some amplitude of the oscillator's response, and ϕ is a phase shift. Perhaps the most-often-overlooked implication is that the steady-state response is periodic, *not* with the oscillator's 'natural period', but instead with the period of the driving waveform.

The most glamorous prediction is that the amplitude A of the response with be 'resonant': that is, that as the driving frequency is varied (with drive amplitude held fixed), A will be a function of ω, with a peak occurring near ω_0. In fact the theory predicts exactly how this peak should look:

$$A(\omega) = A(\omega{=}0)\ \omega_0^2\ [\ (\omega_0^2 - \omega^2)^2 + (2\ \gamma\ \omega\ \omega_0)^2\ ]^{-1/2}\ .$$

This result predicts, for example, that at $\omega = \omega_0$, the amplitude A is enhanced, relative to the 'response for d.c. drive', by the factor $A(\omega_0)/A(0) = 1/(2\gamma)$. This gives the 'Q', or quality factor, of the damped oscillator. By reducing the eddy-current damping to a minimum, the Q can be made to exceed 100 in the TeachSpin oscillator. But since the parameters γ and ω_0 have been previously established by hold-and-release experiments, it's also feasible to explore the full functional dependence of $A(\omega)$, seeing the way this function is peaked in the neighborhood of ω_0 , and finding what 'width' it has.

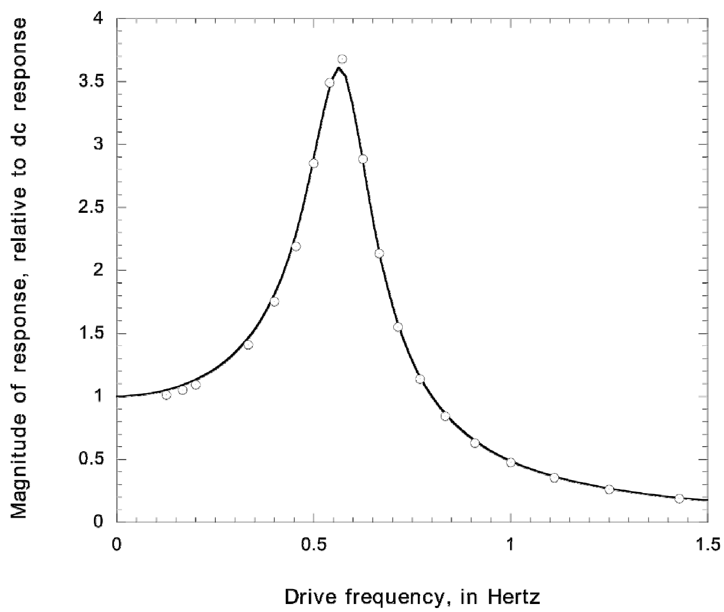

Fig. 3: Magnitude of sinusoidal response per unit magnitude of sinusoidal drive, as a function of the drive frequency chosen. The smooth curve is the predicted result, evaluated with *no free parameters*.

The same method of drive-and-response allows you to see the reality of the phase shift ϕ of the response. This is a less-well-known feature of the driven harmonic oscillator, in that the response is time-delayed, relative to the drive, by a frequency-dependent amount. The qualitative predictions are that

$\phi \approx 0$	for	$\omega << \omega_0$;
$\phi = \pi/2$	for	$\omega = \omega_0$;
$\phi \rightarrow \pi$	for	$\omega >> \omega_0$,

and the detailed quantitative prediction is given by

$$\cos\phi = (\omega_0^2 - \omega^2)\ [\ (\omega_0^2 - \omega^2)^2 + (2\ \gamma\ \omega\ \omega_0)^2\ ]^{-1/2}\ .$$

And this behavior too can be checked: the left-hand-side can be measured by acquiring drive, and response, waveforms, and looking for the time delay, while the right-hand-side is a function which can be predicted in detail as function of ω, using the previously established values of parameters γ and ω_0 .

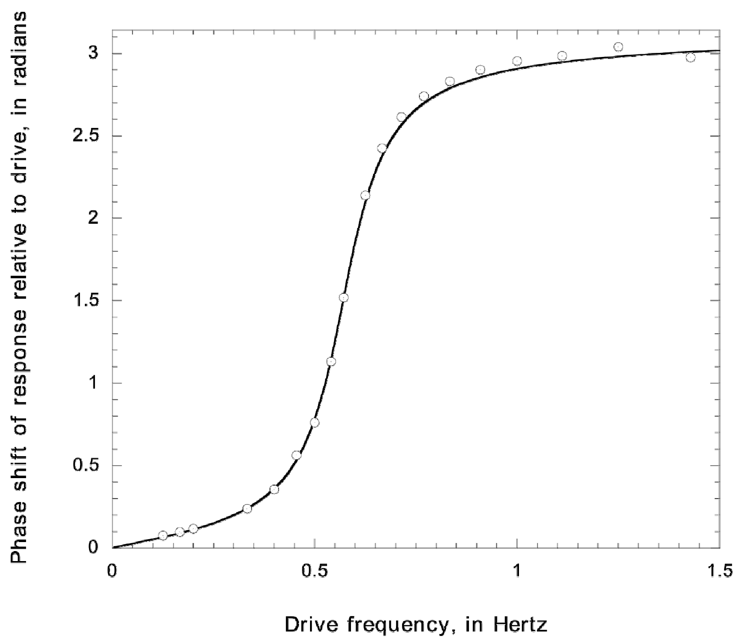

Fig. 4: The phase shift between drive and response waveforms for the sinusoidally driven oscillator. The smooth curve is the predicted result, evaluated with *no free parameters*.

But there's no restriction to sinusoidal drive in this oscillator. Students can easily use triangle, square, arbitrary, or even *noise* waveforms to drive the oscillator. Students can gain lots of intuition about time-domain and frequency-domain viewpoints by exploring the response of this oscillator to various drive waveforms. Best of all, this will develop habits of mind that will be transferable to the behavior of any linear system in any branch of physics.